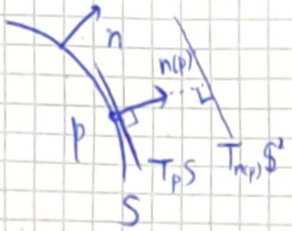
Differential Geometry I $\boxed{\text{Week 11}}$

Let $S \subset \mathbb{R}^3$ be a C^2 surface with a coorientation n.



- Gauss map: $n : S \longrightarrow S^2 \subset \mathbb{R}^3$
- Since we can identify T_pS with $T_{n(p)} S^2$:

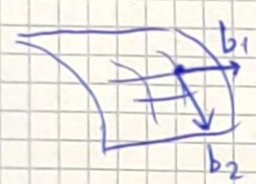
$$\underline{\text{Shape operator}}: \; dn_p : T_pS \longrightarrow T_pS$$

(also L_p)

- Second fundamental form:

$$h_p : T_pS \times T_pS \longrightarrow \mathbb{R}, \quad h_p(v,w) = -\langle L_p(v), w\rangle$$

<u>Proposition</u>: Let $\psi : \Omega \longrightarrow S$ be a C^2 parametrization and let



$b_i = \frac{\partial \psi}{\partial u_i}$, $i = 1,2$, be the corresponding basis vectors of T_pS at $p = \psi(u)$. Then in this basis, the matrix $[h_{ij}] = [h(b_i, b_j)]$ of h_p satisfies

$$h_{ij}(u) = \left\langle n(\psi(u)), \frac{\partial^2 \psi}{\partial u_i \partial u_j}(u) \right\rangle = -\left\langle \frac{\partial}{\partial u_i}(n(\psi(u))), \frac{\partial \psi}{\partial u_j} \right\rangle$$

<u>Corollary</u>: h is symmetric: $h_p(v,w) = h_p(w,v)$ (since $h_p(v,w) = \sum_{i,j} h_{ij}(u) \cdot v^i w^j$ and $h_{ij} = h_{ji}$)

Moreover $\langle L_p(v), w\rangle = \langle v, L_p(w)\rangle$, so L_p is self-adjoint.

<u>Proof</u>: Denote $n(u) := n(\psi(u))$. Then, since $\langle n, b_j\rangle = 0$, by differentiating:

$$\left\langle \frac{\partial n}{\partial u_i}, b_j \right\rangle + \left\langle n, \frac{\partial b_j}{\partial u_i} \right\rangle = 0 \quad ①$$

Note: $\frac{\partial}{\partial u_i} n(u) := \frac{\partial}{\partial u_i}(n(\psi(u))) = dn_{\psi(u)}\left(\frac{\partial \psi}{\partial u_i}(u)\right) = dn_{\psi(u)}(b_i)$

So ① $\Longrightarrow$ $\langle dn_{\psi(u)}(b_i), b_j\rangle + \left\langle n, \frac{\partial b_j}{\partial u_i}\right\rangle = 0$

So $h_p(b_i, b_j) = -\langle L_p(b_i), b_j\rangle = -\langle dn_{\psi(u)}(b_i), b_j\rangle = \left\langle n, \frac{\partial b_i}{\partial u_j}\right\rangle =$

$= \langle n, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}\psi \rangle$. □

<u>Proposition</u>: In the basis $\{b_1, b_2\}$ of T_pS (for any $p \in S$), Let H be the matrix of h, G the first fundamental form and L the matrix of L_p. Then

$$H = -GL = -L^TG$$

$$(\Leftrightarrow L = -G^{-1}H)$$

<u>Proof</u>:

For any $\xi, \eta \in T_pS$: If $\xi = \xi_1 b_1 + \xi_2 b_2$ (so $\xi = \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$) and $\eta = \eta_1 b_1 + \eta_2 b_2$, then $h(\xi,\eta) = \xi^T H \eta$, $\langle \xi, \eta \rangle = \xi^T G \eta$.

(Since $h(\xi,\eta) = h(\sum_{i=1}^{2} \xi_i b_i, \sum_{j=1}^{2} \eta_j b_j) = \sum_{i=1}^{2}\sum_{j=1}^{2} h(b_i, b_j)\xi_i \eta_j$.)

So: $\xi^T H \eta = h(\xi,\eta) = -\langle L(\xi), \eta \rangle = -(L\xi)^T G \eta = -\xi^T L^T G \eta$

True for any $\xi, \eta \Rightarrow H = -L^TG$. Since h symmetric:

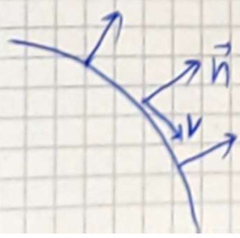
$H = H^T = (-L^TG)^T = -G^TL = -GL$ ($G^T = G$) □

<u>Curvature of a surface</u>:

Let S be a C^2 co-oriented surface and $p \in S$.

<u>Def</u>: $\forall v \in T_pS \setminus \{0\}$: <u>Normal curvature</u> in the direction of v:

$$k_n(v) = \frac{h_p(v,v)}{\langle v,v \rangle} = -\frac{\langle L_p(v), v\rangle}{\langle v,v\rangle}$$

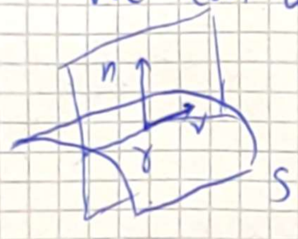


← In this case: $k_n(v) < 0$

(Different authors: Different sign convention)

Note: From Meusnier's theorem: If $\gamma: I \to S$ is any curve on S such that $\gamma(t) = p$, then the normal curvature of γ at $\gamma(t)$ is equal to the normal curvature of S at p in the direction $\dot{\gamma}(t)$. Hence we use the same symbol.

We can always choose as the curve γ:



The intersection of S with the plane spanned by $\{n, v\}$ (normal section)

Then $k_n(p)$ = curvature of that planar curve

Let us consider the Weingarten map (or shape operator)

$$L_p : T_pS \to T_pS.$$

It is a self-adjoint operator $\left(\langle L_p(\xi), \eta \rangle = \langle \xi, L_p(\eta) \rangle \ \forall \xi, \eta \in T_pS \right)$

therefore from the spectral theorem:

- It has orthonormal eigenvectors $\{e_1, e_2\}$
- The corresponding eigenvalues $-k_1, -k_2$ are real

Definition: The eigenvalues k_1, k_2 of $-L_p$: Principal curvatures (ordered so that $k_1 \leq k_2$). So $k_1, k_2 : S \longrightarrow \mathbb{R}$.

Definition: Gauss curvature of S at p:

$$K(p) = \det(L_p) = k_1(p)\, k_2(p).$$

We use the following terminology:

Def: The point p of the surface S is

- Elliptic, if $K(p) > 0$ (so k_1, k_2: same sign; h is either positive definite or negative definite)
- Hyperbolic if $K(p) < 0$ (so k_1, k_2: opposite sign)
- Parabolic if $k_1 = 0$ or $k_2 = 0$
- Planar if $k_1 = k_2 = 0$
- Umbilic if $k_1 = k_2$ (in this case: $L_p = \lambda \cdot \mathrm{I}$)

Note: If we flip the co-orientation: $\nu \longrightarrow -\nu$ then we have $L_p \to -L_p$, $h_p \to -h_p$ (so the signs of k_1, k_2 change) but $K(p)$ stays the same, and similarly the classification of a point p as above stays the same.

Def: • Mean curvature at p: $H = \frac{1}{2}(k_1 + k_2)$ (so changes sign when flipping co-orientation)

- If p is not umbilic: Principal direction: The unique eigenvectors of L_p, e_1, e_2 normalized to have unit length. (If p is umbilic: Any ~~orthonormal~~ vector in T_pS is an eigenvector)

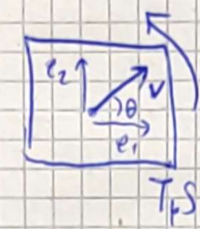
- A curve γ on S is a line of curvature if $\dot{\gamma}(t)$ is parallel to a principal direction at every point.

• Note: Since $H = -GL$, we have

$$K(p) = \det(L_p) = \frac{\det(H(p))}{\det(G(p))}.$$

Suppose that at $p \in S$, we look at the family of normal sections obtained by rotating the intersecting plane:

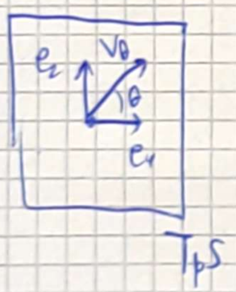
$\longrightarrow$



How does the curvature of the normal section change?

Theorem (Euler): Let $\{e_1, e_2\}$ be the principal direction at T_pS (if p is non-umbilic), or any orthonormal basis of T_pM if p is umbilic. If $v_\theta = \cos\theta \cdot e_1 + \sin\theta \cdot e_2$, then

$$\boxed{k_n(v_\theta) = k_1 \cdot \cos^2\theta + k_2 \cdot \sin^2\theta}$$



Corollaries:
- k_1 is the minimum value of $k_n(v_\theta)$, k_2 is the maximum value as we rotate v_θ
- If $k_1, k_2 > 0$ $k_n(v) > 0$ always

Proof: Since $\langle v_\theta, v_\theta \rangle = 1$, we have

$$k_n(v_\theta) = \frac{h_p(v_\theta, v_\theta)}{\langle v_\theta, v_\theta \rangle} = h_p(\cos\theta \cdot e_1 + \sin\theta \cdot e_2, \cos\theta \cdot e_1 + \sin\theta \cdot e_2)$$
$$= \cos^2\theta \cdot h(e_1, e_1) + 2\cos\theta \sin\theta \cdot h(e_1, e_2) + \sin^2\theta \cdot h(e_2, e_2)$$

Since $L_p(e_1) = -k_1 e_1$, $L_p(e_2) = -k_2 e_2$:

$$h_p(e_1, e_1) = -\langle L_p(e_1), e_1 \rangle = k_1 \langle e_1, e_1 \rangle = k_1$$
$$h_p(e_1, e_2) = -\langle L_p(e_1), e_2 \rangle = k_1 \langle e_1, e_2 \rangle = 0.$$
$$h_p(e_2, e_2) = k_2.$$
$\boxtimes$

Proof of corollary

$$k_n(v_\theta) = \cos^2\theta \cdot k_1 + \sin^2\theta \cdot k_2 = k_1 + (k_2 - k_1) \cdot \sin^2\theta.$$

since $0 \leq \sin^2\theta \leq 1$: we get the result.
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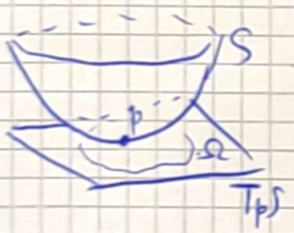
How does the shape of a surface look like around an elliptic / hyperbolic / parabolic point?

It is convenient to represent S as the graph of a function



If $p \in S$: I can always find Cartesian coordinates (x,y,z) so that
$p = (0,0,0)$ and $T_pS = \{z = 0\}$
Then locally around p: S can be represented as the graph of a C^2 function:
$$z = \phi(x,y) \quad \text{for } (x,y) \in \Omega \subseteq \mathbb{R}^2 \equiv T_pS$$

- $p = (0,0,0) \Rightarrow \phi(0,0) = 0$
- $T_pS = \{z = 0\} \Rightarrow \frac{\partial \phi}{\partial x}(0,0) = \frac{\partial \phi}{\partial x}(0,0) = 0.$

Parametrization: $\psi : \Omega \longrightarrow S \subset \mathbb{R}^3$, $\psi(x,y) = (x, y, \phi(x,y))$

and $b_1 = \frac{\partial \psi}{\partial x}(0) = (1,0,0) = e_1$
$b_2 = \frac{\partial \psi}{\partial y}(0) = (0,1,0) = e_2$

Choose co-orientation:

$n = b_1 \times b_2 = e_3$ at p.

- Metric tensor $G_p = \begin{bmatrix} \langle b_1, b_1\rangle_p & \langle b_1, b_2\rangle_p \\ \cdots & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Second fundamental form: h_p:

$$h_{11} = \langle n, \frac{\partial^2 \varphi}{\partial x^2} \rangle = \langle e_3, \frac{\partial^2 \varphi}{\partial x^2} \rangle = \frac{\partial^2 \phi}{\partial x^2}(0)$$

$$h_{12} = \frac{\partial^2 \phi}{\partial x \partial y}(0), \quad h_{22} = \frac{\partial^2 \phi}{\partial y^2}(0)$$

So $H_p = \begin{bmatrix} \phi_{xx}(0) & \phi_{xy}(0) \\ \phi_{xy}(0) & \phi_{yy}(0) \end{bmatrix}$

- Shape operator: $L_p = -G_p H_p = -\begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{bmatrix}$

- Gauss curvature: $K = \det L = \phi_{xx}(0)\,\phi_{yy}(0) - (\phi_{xy}(0))^2$

- Mean curvature: $H = -\frac{1}{2} \operatorname{trace} L = \frac{1}{2}(\phi_{xx}(0) + \phi_{yy}(0))$

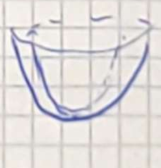
By rotating in the plane I can make $\{e_1, e_2\}$ be principal directions. Then L_p: <u>diagonal</u> (so $\phi_{xy} = 0$)

Taylor expansion for ϕ: In this case (since $\phi_{xy}(0) = 0$)

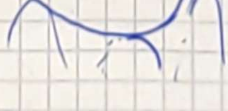
$$\phi(x,y) = \frac{1}{2}(ax^2 + by^2) + o(x^2+y^2)$$

So $k_1(0) = a$, $k_2(0) = b$, $K = ab$

- Elliptic point: $a, b > 0$ (or both < 0):



- Hyperbolic point: $a < 0 < b$



- Parabolic point: $a = 0$ or $b = 0$:

Note: Using our sign convention:

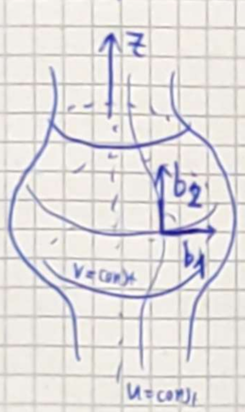
If S is convex and n points inwards, then h is positive definite.

(Sometimes different authors use the opposite sign convention, i.e. $h(v,w) = +\langle L_p(v), w\rangle$)

- On a sphere or on a plane: Every point is umbilic (we will show that's an if and only if statement).
- If S is contained in the boundary of a convex domain: $K \geq 0$ (locally. The converse also true).

Let's calculate the curvature for some explicit examples:

Let $\alpha(v) = (r(v), z(v))$ $(v \in I)$ be a C^2 curve in the plane, $r(v) > 0$, and let S be the surface of revolution obtained by rotating α around the z axis. Assume $\|\dot\alpha\| = 1$.



So S parametrized by

$$\psi : [0, 2\pi) \times I \longrightarrow S,$$

$$\psi(u,v) = (r(v)\cdot \cos u,\ r(v)\cdot \sin u,\ z(v))$$

Basis for $T_{\psi(u,v)} S$: $b_1 = \frac{\partial \psi}{\partial u} = \begin{bmatrix} -r(v)\cdot \sin u \\ r(v)\cdot \cos u \\ 0 \end{bmatrix}$, $b_2 = \frac{\partial \psi}{\partial v} = \begin{bmatrix} \dot r(v)\cdot \cos u \\ \dot r(v)\cdot \sin(u) \\ \dot z(v) \end{bmatrix}$

$$n = \frac{b_1 \times b_2}{\|b_1 \times b_2\|} = \begin{bmatrix} \dot z(v)\cdot \cos u \\ \dot z(v)\cdot \sin u \\ -\dot r(v) \end{bmatrix}$$

(we used that $\|\dot\alpha\| = 1 \Leftrightarrow \dot r^2 + \dot z^2 = 1$)

Then

- Metric tensor: $G = \begin{bmatrix} r^2(v) & 0 \\ 0 & 1 \end{bmatrix}$

- $\dfrac{\partial^2 \psi}{\partial u^2} = \begin{bmatrix} -r(v)\cdot\cos u \\ -r(v)\cdot\sin u \\ 0 \end{bmatrix}$, $\dfrac{\partial^2 \psi}{\partial u \partial v} = \begin{bmatrix} -\dot r(v)\cdot\sin(u) \\ \dot r(v)\cdot\cos(u) \\ 0 \end{bmatrix}$, $\dfrac{\partial^2 \psi}{\partial v^2} = \begin{bmatrix} \ddot r(v)\cdot\cos u \\ \ddot r(v)\cdot\sin u \\ \ddot z(v) \end{bmatrix}$

So $h_{ij} = \left\langle n, \dfrac{\partial^2 \psi}{\partial u_i \partial u_j} \right\rangle \Rightarrow H = \begin{bmatrix} -r(v)\cdot\dot z(v) & 0 \\ 0 & \ddot r(v)\cdot\dot z(v) - \ddot z(v)\,\dot r(v) \end{bmatrix}$

- Gauss curvature: $K = \dfrac{\det H}{\det G} = \dfrac{-r\cdot\dot z(\ddot r\cdot\dot z - \ddot z\cdot\dot r)}{r^2} = \dfrac{\dot z\,\ddot z\cdot\dot r - (\dot z)^2\cdot\ddot r}{r}$

Since $\dot r^2 + \dot z^2 = 1 \Longrightarrow \dot r\cdot\ddot r + \dot z\cdot\ddot z = 0$

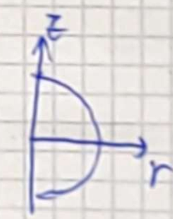
We have $\dot z^2 = 1 - \dot r^2$
and $\dot z\,\ddot z = -\dot r\cdot\ddot r$ $\Bigg\}$ $\boxed{K = -\dfrac{1}{r(v)}\cdot\ddot r(v)}$ (depends only on v! Rotation in u is isometry)

- Weingarten map:

$L = -G^{-1}H = -\begin{bmatrix} r^{-2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -r\cdot\dot z & 0 \\ 0 & \ddot r\cdot\dot z - \ddot z\cdot\dot r \end{bmatrix} = \begin{bmatrix} \frac{\dot z}{r} & 0 \\ 0 & \ddot z\cdot\dot r - \ddot r\cdot\dot z \end{bmatrix}$

So: - b_1, b_2: <u>principal directions</u> (so meridians and parallels are lines of curvature)

- For a sphere: Obtained by rotating the curve



$r(v) = R\cdot\cos\left(\frac{v}{R}\right)$, $z(v) = R\cdot\sin\left(\frac{v}{R}\right)$, $v \in \left(-\frac{\pi}{2R}, \frac{\pi}{2R}\right)$

Then $L = \begin{bmatrix} 1/R & 0 \\ 0 & 1/R \end{bmatrix}$ So every point is <u>umbilic</u> and $\boxed{K = \dfrac{1}{R^2}}$

What if a surface of revolution has constant curvature?

If $K = \text{const}$, then from $K = -\frac{1}{r} \cdot \ddot{r} \Rightarrow \ddot{r} + Kr = 0$

From unit speed: $\dot{z} = \sqrt{1 - \dot{r}^2}$

So we can determine the curve α (and hence the surface!)

- If $K = +1$: $\ddot{r} + r = 0$ One solution: $r(v) = \cos v$

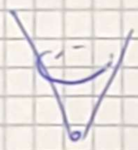
$$\Rightarrow z(v) = \sin v$$

So: the sphere

(any other solution would just give a rotated parametrization of the sphere)

- If $K = 0$: $\ddot{r} = 0 \Rightarrow r(v) = a \cdot v$

and $z(v) = \sqrt{1 - a^2} \cdot v$



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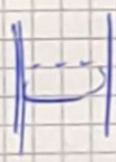

- If $K = -1$: One solution

$$r(v) = e^{-v}, \quad z(v) = \int_0^v \sqrt{1 - e^{-2s}}\, ds$$

(Even without the assumption that S is rotationally symmetric: $K = +1$ implies that each connected component of S is part of a sphere of radius 1)

<u>Theorem</u>: If all the points of S (assuming it's connected and of class C^3) are umbilic, then S is contained in a sphere or a plane

<u>Proof</u>: Assume that $\forall p \in S$, $L_p : T_pS \to T_pS$ is of the form $L_p = \lambda(p) \cdot Id$ (i.e. p is umbilic).

Let $\psi: \Omega \to S$ be a local parametrization of S, then

$\forall u \in \Omega$: (denoting $n(u) := n(\psi(u))$ and $\lambda(u) := \lambda(\psi(u))$):

$$\underbrace{L_{\psi(u)}(b_i)}_{dn_{\psi(u)}(b_i)} = \lambda(u)\, b_i \iff \frac{\partial}{\partial u_i}(n(u)) = \lambda(u) \cdot \frac{\partial \psi}{\partial u_i}$$

So differentiating with $\frac{\partial}{\partial u_j}$:

$$\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_i} n(u) = \frac{\partial \lambda}{\partial u_j} \cdot \frac{\partial \psi}{\partial u_i} + \lambda \cdot \frac{\partial^2 \psi}{\partial u_j \partial u_i}$$

Doing the same but with $i \leftrightarrow j$:

$$\frac{\partial^2}{\partial u_i \partial u_j} n(u) = \frac{\partial^2}{\partial u_j \partial u_i} n(u) \iff \frac{\partial \lambda}{\partial u_i} \cdot \frac{\partial \psi}{\partial u_j} + \cancel{\lambda \frac{\partial^2 \psi}{\partial u_i \partial u_j}} = \frac{\partial \lambda}{\partial u_j} \cdot \frac{\partial \psi}{\partial u_i} + \cancel{\lambda \frac{\partial^2 \psi}{\partial u_i \partial u_j}}$$

$i=1, j=2$: Since $\frac{\partial \psi}{\partial u_1}, \frac{\partial \psi}{\partial u_2}$ linearly independent $\Rightarrow \frac{\partial \lambda}{\partial u_1} = \frac{\partial \lambda}{\partial u_2} = 0$

So $\lambda = \text{const}$ on S (since S is connected)

- If $\lambda = 0$: $L_p = 0$ $\forall p \in S \Rightarrow dn_p = 0$ $\forall p$ $\Rightarrow n = \text{const}$ So S is contained in a plane
- If $\lambda \neq 0$: $\psi(u) - \frac{1}{\lambda} \cdot n(u)$ is constant on S so S inside the sphere centered at $\psi - \frac{1}{\lambda} n$ of radius $\frac{1}{|\lambda|}$.

$\boxed{\ }$